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Using Trigonometric Substitution Method to Solve Some Fractional Integral Problems

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Abstract: **In this paper, based on Jumarie's modified Riemann Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study some fractional integral problems. The main methods we used are the trigonometric substitution method and change of variables for fractional calculus. In fact, these results we obtained are generalizations of those in traditional calculus. On the other hand, some examples are provided to illustrate how to use these methods to evaluate the fractional integrals.**

Keywords: **Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, fractional integral, trigonometric substitution method, change of variables.**

I. INTRODUCTION

Fractional calculus is a natural extension of classical calculus, which has a history of more than 300 years. In fact, since the birth of differential and integral theory, several mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. However, although much work has been done, the application of fractional derivatives and integrals has only recently begun. In recent years, the development of fractional calculus has stimulated people's new interest in physics, engineering, economics, biology and other scientific fields [1-8].

This paper studies some fractional integral problems based on Jumarie type of R-L fractional calculus. A new multiplication of fractional analytic functions plays an important role in this article. The main methods we used are the trigonometric substitution method and change of variables for fractional calculus. In fact, these results we obtained are generalizations of classical calculus results. In addition, we give some example to illustrate how to use these methods to calculate the fractional integrals.

II. DEFINITIONS AND PROPERTIES

First, the fractional calculus used in this paper and its some properties are introduced.

Definition 2.1 ([9])**:** Assume that $0 < \alpha \leq 1$, and t_0 is a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}\frac{f(x)-f(t_0)}{(t-x)^{\alpha}}dx. \quad (1)$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{t_0}I_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(x)}{(t-x)^{1-\alpha}}dx, \quad (2)$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([10])**:** *Suppose that* α, β, t_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{t_0}D_t^{\alpha}\right)\left[(t-t_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha}, \quad (3)$$

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and

$$\left({}_{t_0}D_t^{\alpha}\right)[C]=0. \tag{4}$$

Next, we introduce the fractional analytic function.

Definition 2.3 ([11])**:** Let t, t_0, and a_k be real numbers for all k, $t_0 \in (a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as $f_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}$, an α-fractional power series on some open interval containing t_0, then we say that $f_\alpha(t^\alpha)$ is α-fractional analytic at t_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([12])**:** If $0<\alpha\leq 1$, and t_0 is a real number. Let $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ be two α-fractional analytic functions defined on an interval containing t_0 ,

$$f_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned}&f_\alpha(t^\alpha)\otimes g_\alpha(t^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(t-t_0)^{k\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(t^\alpha)\otimes g_\alpha(t^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}.\end{aligned} \tag{8}$$

Definition 2.5 ([13])**:** Let $0<\alpha\leq 1$, and $f_\alpha(t^\alpha)$, $g_\alpha(t^\alpha)$ be two α-fractional analytic functions defined on an interval containing t_0 ,

$$f_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(t^\alpha)=f_\alpha(g_\alpha(t^\alpha))=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(t^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(t^\alpha)=g_\alpha(f_\alpha(t^\alpha))=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(f_\alpha(t^\alpha)\right)^{\otimes k}. \tag{12}$$

Definition 2.6 ([13])**:** Let $0<\alpha\leq 1$. If $f_\alpha(t^\alpha)$, $g_\alpha(t^\alpha)$ are two α-fractional analytic functions at t_0 satisfies

$$(f_\alpha \circ g_\alpha)(t^\alpha) = (g_\alpha \circ f_\alpha)(t^\alpha) = \frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha. \tag{13}$$

Then $f_\alpha(t^\alpha)$, $g_\alpha(t^\alpha)$ are called inverse functions of each other.

Next, The followings are some fractional analytic functions.

Definition 2.7([14])**:** If $0 < \alpha \le 1$, and t is a real number. The α-fractional exponential function is defined by

$$E_\alpha(t^\alpha) = \sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes k}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(t^\alpha)$ is the inverse function of $E_\alpha(t^\alpha)$. In addition, the α-fractional cosine and sine function are defined respectively as follows:

$$cos_\alpha(t^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2k}, \tag{15}$$

and

$$sin_\alpha(t^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes(2k+1)}. \tag{16}$$

On the other hand,

$$sec_\alpha(t^\alpha) = \left(cos_\alpha(t^\alpha)\right)^{\otimes -1} \tag{17}$$

is called the α-fractional secant function.

$$csc_\alpha(t^\alpha) = \left(sin_\alpha(t^\alpha)\right)^{\otimes -1} \tag{18}$$

is the α-fractional cosecant function.

$$tan_\alpha(t^\alpha) = sin_\alpha(t^\alpha) \otimes sec_\alpha(t^\alpha) \tag{19}$$

is the α-fractional tangent function. And

$$cot_\alpha(t^\alpha) = cos_\alpha(t^\alpha) \otimes csc_\alpha(t^\alpha) \tag{20}$$

is the α-fractional cotangent function.

In the following, inverse fractional trigonometric functions are introduced.

Definition 2.8 [18]**:** Let $0 < \alpha \le 1$. Then $arcsin_\alpha(t^\alpha)$ is the inverse function of $sin_\alpha(t^\alpha)$, and it is called inverse α-fractional sine function. $arccos_\alpha(t^\alpha)$ is the inverse function of $cos_\alpha(t^\alpha)$, and we say that it is the inverse α-fractional cosine function. On the other hand, $arctan_\alpha(t^\alpha)$ is the inverse function of $tan_\alpha(t^\alpha)$, and it is called the inverse α-fractional tangent function. $arccotan_\alpha(t^\alpha)$ is the inverse function of $cot_\alpha(t^\alpha)$, and we say that it is the inverse α-fractional cotangent function. $arcsec_\alpha(t^\alpha)$ is the inverse function of $sec_\alpha(t^\alpha)$, and it is the inverse α-fractional secant function. $arccsc_\alpha(t^\alpha)$ is the inverse function of $csc_\alpha(t^\alpha)$, and is called the inverse α-fractional cosecant function.

Definition 2.9 [15]**:** Let $0 < \alpha \le 1$, and s be a real number. The s-th power of the α-fractional analytic function $f_\alpha(t^\alpha)$ is defined by $[f_\alpha(t^\alpha)]^{\otimes s} = E_\alpha\left(s \cdot Ln_\alpha\left(f_\alpha(t^\alpha)\right)\right)$.

Definition 2.10: The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(it^\alpha)$.

III. METHODS AND EXAMPLES

In the following, we introduce some properties used in this article and provide several examples to illustrate how to use trigonometric substitution method to evaluate some fractional integrals.

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Theorem 3.1 (change of variables for fractional calculus)[19]**:** *Suppose that* $0 < \alpha \leq 1$, $w_\alpha(t^\alpha)$ *is an* α*-fractional analytic function defined on an interval* I, *and* $f_\alpha\big(w_\alpha(t^\alpha)\big)$ *is an* α*-fractional analytic function such that the range of* w_α *contained in the domain of* f_α , *then*

$$\left({}_{w_\alpha(c^\alpha)}I^{\alpha}_{w_\alpha(d^\alpha)}\right)[f_\alpha(w_\alpha)] = \left({}_cI^{\alpha}_d\right)\Big[f_\alpha\big(w_\alpha(t^\alpha)\big) \otimes \left({}_cD^{\alpha}_t\right)[w_\alpha(t^\alpha)]\Big], \tag{21}$$

for $c, d \in I$.

Theorem 3.2 [17]**:** *Let* $0 < \alpha \leq 1$, $q > 0$, $sin_\alpha(q^\alpha) \neq 0$, *and* t *be a real number. Then*

$$\left({}_0I^{\alpha}_t\right)[sin_\alpha(t^\alpha)] = -cos_\alpha(t^\alpha) + 1, \text{ if } t \geq 0. \tag{22}$$

$$\left({}_0I^{\alpha}_t\right)[cos_\alpha(t^\alpha)] = sin_\alpha(t^\alpha), \text{ if } t \geq 0. \tag{23}$$

$$\left({}_0I^{\alpha}_t\right)[tan_\alpha(t^\alpha)] = -Ln_\alpha(|cos_\alpha(t^\alpha)|), \text{ if } t \geq 0. \tag{24}$$

$$\left({}_qI^{\alpha}_t\right)[cot_\alpha(t^\alpha)] = Ln_\alpha(|sin_\alpha(t^\alpha)|) - Ln_\alpha(|sin_\alpha(q^\alpha)|), \text{ if } t \geq q. \tag{25}$$

$$\left({}_0I^{\alpha}_t\right)[sec_\alpha(t^\alpha)] = Ln_\alpha(|sec_\alpha(t^\alpha) + tan_\alpha(t^\alpha)|), \text{ if } t \geq 0. \tag{26}$$

$$\left({}_qI^{\alpha}_t\right)[csc_\alpha(t^\alpha)] = Ln_\alpha(|csc_\alpha(t^\alpha) - cot_\alpha(t^\alpha)|) - Ln_\alpha(|csc_\alpha(q^\alpha) - cot_\alpha(q^\alpha)|), \text{ if } t \geq q. \tag{27}$$

Theorem 3.3 ([16])**:** *Let* $0 < \alpha \leq 1$, *and* t *be a real number, then*

$$[sin_\alpha(t^\alpha)]^{\otimes 2} + [cos_\alpha(t^\alpha)]^{\otimes 2} = 1, \tag{28}$$

$$1 + [tan_\alpha(t^\alpha)]^{\otimes 2} = [sec_\alpha(t^\alpha)]^{\otimes 2}, \tag{29}$$

$$1 + [cot_\alpha(t^\alpha)]^{\otimes 2} = [csc_\alpha(t^\alpha)]^{\otimes 2}. \tag{30}$$

Example 3.4: Let $0 < \alpha \leq 1$, $r > 0$, and $0 \leq x \leq \big(r\Gamma(\alpha+1)\big)^{\frac{1}{\alpha}}$. Find the α-fractional integral

$$\left({}_0I^{\alpha}_x\right)\left[\left(r^2 - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\left(\frac{1}{2}\right)}\right]. \tag{31}$$

Solution By trigonometric substitution method, we let $\frac{1}{\Gamma(\alpha+1)}x^\alpha = r \cdot sin_\alpha(t^\alpha)$, then using change of variables for fractional calculus,

$$\left({}_0I^{\alpha}_x\right)\left[\left(r^2 - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\left(\frac{1}{2}\right)}\right]$$

$$= \left({}_0I^{\alpha}_t\right)[r \cdot cos_\alpha(t^\alpha) \otimes r \cdot cos_\alpha(t^\alpha)]$$

$$= r^2\left({}_0I^{\alpha}_t\right)\left[\left(cos_\alpha(t^\alpha)\right)^{\otimes 2}\right]$$

$$= r^2\left({}_0I^{\alpha}_t\right)\left[\frac{1}{2} + \frac{1}{2}cos_\alpha(2t^\alpha)\right]$$

$$= r^2\left[\frac{1}{2} \cdot \frac{1}{\Gamma(\alpha+1)}t^\alpha + \frac{1}{4}sin_\alpha(2t^\alpha)\right]$$

$$= r^2\left[\frac{1}{2} \cdot \frac{1}{\Gamma(\alpha+1)}t^\alpha + \frac{1}{2}sin_\alpha(t^\alpha) \otimes cos_\alpha(t^\alpha)\right]$$

$$= r^2\left[\frac{1}{2}arcsin_\alpha\left(\frac{1}{r}x^\alpha\right) + \frac{1}{2r^2} \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes \left(r^2 - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\left(\frac{1}{2}\right)}\right]$$

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$$=\frac{1}{2}r^2\cdot arcsin_\alpha\left(\frac{1}{r}x^\alpha\right)+\frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\left(r^2-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes\left(\frac{1}{2}\right)}. \quad (32)$$

Example 3.5: If $0<\alpha\leq 1$, $r>0$, and $x\geq 0$. Find the α-fractional integral

$$\left({}_0I_x^\alpha\right)\left[\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+r^2\right)^{\otimes\left(-\frac{1}{2}\right)}\right]. \quad (33)$$

Solution Let $\frac{1}{\Gamma(\alpha+1)}x^\alpha=r\cdot tan_\alpha(t^\alpha)$, then by change of variables for fractional calculus, we obtain

$$\left({}_0I_x^\alpha\right)\left[\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+r^2\right)^{\otimes\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_0I_t^\alpha\right)\left[\frac{1}{r}\cdot\left(sec_\alpha(t^\alpha)\right)^{\otimes(-1)}\otimes r\cdot\left(sec_\alpha(t^\alpha)\right)^{\otimes 2}\right]$$

$$=\left({}_0I_t^\alpha\right)\left[sec_\alpha(t^\alpha)\right]$$

$$=Ln_\alpha\left(\left|sec_\alpha(t^\alpha)+tan_\alpha(t^\alpha)\right|\right)$$

$$=Ln_\alpha\left(\left|\frac{1}{r}\cdot\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+r^2\right)^{\otimes\left(\frac{1}{2}\right)}+\frac{1}{r}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right|\right)$$

$$=Ln_\alpha\left(\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+r^2\right)^{\otimes\left(\frac{1}{2}\right)}+\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)-Ln_\alpha(r). \quad (34)$$

Example 3.6: If $0<\alpha\leq 1$, $r>0$, and $x\geq p>\left(r\Gamma(\alpha+1)\right)^{\frac{1}{\alpha}}$. Find the α-fractional integral

$$\left({}_pI_x^\alpha\right)\left[\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-r^2\right)^{\otimes\left(-\frac{1}{2}\right)}\right]. \quad (35)$$

Solution Let $\frac{1}{\Gamma(\alpha+1)}x^\alpha=r\cdot sec_\alpha(t^\alpha)$, and $q=\left[\Gamma(\alpha+1)\cdot arcsec_\alpha\left(\frac{1}{r}p^\alpha\right)\right]^{\frac{1}{\alpha}}$, then using change of variables for fractional calculus yields

$$\left({}_pI_x^\alpha\right)\left[\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-r^2\right)^{\otimes\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_qI_t^\alpha\right)\left[\frac{1}{r}\cdot\left(tan_\alpha(t^\alpha)\right)^{\otimes(-1)}\otimes r\cdot sec_\alpha(t^\alpha)\otimes tan_\alpha(t^\alpha)\right]$$

$$=\left({}_qI_t^\alpha\right)\left[sec_\alpha(t^\alpha)\right]$$

$$=Ln_\alpha\left(\left|sec_\alpha(t^\alpha)+tan_\alpha(t^\alpha)\right|\right)-Ln_\alpha\left(\left|csc_\alpha(q^\alpha)-cot_\alpha(q^\alpha)\right|\right)$$

$$=Ln_\alpha\left(\left|\frac{1}{r}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+\frac{1}{r}\cdot\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-r^2\right)^{\otimes\left(\frac{1}{2}\right)}\right|\right)-Ln_\alpha\left(\left|csc_\alpha(q^\alpha)-cot_\alpha(q^\alpha)\right|\right)$$

$$=Ln_\alpha\left(\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-r^2\right)^{\otimes\left(\frac{1}{2}\right)}+\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)-Ln_\alpha\left(\left(\left(\frac{1}{\Gamma(\alpha+1)}p^\alpha\right)^{\otimes 2}-r^2\right)^{\otimes\left(\frac{1}{2}\right)}+\frac{1}{\Gamma(\alpha+1)}p^\alpha\right). \quad (36)$$

IV. CONCLUSION

The purpose of this paper is to study some fractional integral problems. The main methods we used are the trigonometric substitution method and change of variables for fractional calculus based on Jumarie modification of R-L fractional calculus. A new multiplication plays an important role in this article. In fact, the results we obtained are natural generalizations of those in classical calculus. On the other hand, the new multiplication we defined is a natural operation of fractional analytic functions. In the future, we will use these methods to solve the problems in engineering mathematics and fractional differential equations.

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